GROUPS WHOSE PROPER FACTORS ARE ALL ABELIAN

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ABSTRACT

The question implied in the title is a problem of A. Zaks, namely, which finite groups (other than abelian and simple groups) have all their proper factors abelian? This paper answers the question in the case of groups with non-trivial centre, or, equivalently, in the case of p-groups, and gives a structure theorem for such groups.

DEFINITION. We will call a finite non-abelian group G a *Z-group* if all its proper factors are abelian, and a *Z-p-group* if G is also a p-group, i.e. has order a power of a prime number p.

It transpires that the class of Z-p-groups contains an apparently wider class of Z-groups:

THEOREM 1. *Let G be a finite Z-group with non-trivial centre. Then G is a p-group.*

PROOF. If $Z(G)$ is non-trivial, $G/Z(G)$ is abelian, so that $G' \subset Z(G)$. Thus G is nilpotent, so that all its Sylow-p subgroups are normal. Since the intersection of Sylow subgroups corresponding to different primes is trivial, and G', which is non-trivial, is contained in the intersection of all non-trivial normal subgroups of G, G can have only one non-trivial Sylow subgroup.

Now we have a simple criterion for determining if a p-group is a Z-p-group. Let $Z(G)$ denote the centre of a group G, and G' the derived subgroup of G. Then

THEOREM 2. *Let G be a finite non-abelian p-group. G is a Z-group if and only if $Z(G)$ is cyclic, and contains G' which is of order p.*

PROOF. Suppose $Z(G)$ is cyclic, $|G'| = p$, and $G' \subset Z(G)$. Let N be a non-

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trivial normal subgroup of G. If $N \subset Z(G)$, then $N \cap G' \neq \{1\}$, so that $N \supset G'$, since G' is simple, and therefore G/N is abelian. If on the other hand $N \not\subset Z(G)$, then there are elements $g \in G$ and $n \in N$ for which the commutator $[n, g] \neq 1$. But $[n, g] \in N$ since $N \lhd G$, so that again $N \cap G' \neq \{1\}$, and G/N is abelian.

Conversely, if G is a Z-group, $Z(G) \neq \{1\}$ since G is a finite p-group, and so $G/Z(G)$ is abelian. Thus $G' \subset Z(G)$. Further, G' is contained in all non-trivial normal subgroups of G, and, in particular, G' is contained in the intersection of all subgroups of $Z(G)$. This intersection is thus non-trivial, since G is not abelian, so that by the basis theorem for abelian groups, $Z(G)$ must be cyclic. Since $Z(G)$ contains a subgroup of order p, that subgroup must be G'.

We use this theorem to show that in certain circumstances the product of Z-groups is a Z-group.

THEOREM 3. *Suppose G is a finite p-group, with normal Z-subgroups A, B satisfying*

$$G = AB, \ [A, B] = \{1\}, \ \textit{and} \ Z(A) \subset Z(B).$$

Then G is a Z-group. Moreover, the same conclusion holds if B is cyclic.

PROOF. We must prove that $Z(G)$ is cyclic, and contains G' of order p. Since $[A, B] = 1$, we must have both

$$Z(G) = Z(A)Z(B) \text{ and } G' = A'B'.$$

Now $Z(A) \subset Z(B)$, so that $Z(G) = Z(B)$ which is cyclic. A is a Z-group therefore A' is of order p, and $A' \subset Z(A) \subset Z(G)$. Finally, B is a Z-group, or cyclic, so that B' is of order p, or 1, and contained in $Z(B)$. Now $Z(B)$ has a unique subgroup of order p, namely A', so that $B' \subset A'$, and $G' = A'B' = A'$.

DEFINITION. Let G be a finite p-group. The number of elements in a minimal basis of G, that is the smallest number of elements generating G, is the *rank* of G, which we will denote $r(G)$. [1, p. 35].

THEOREM 4. *Let G be a Z-p-group of rank $s > 2$. Then there exist normal subgroups A, B of G, of ranks 2, $s - 2$ respectively, such that*

$$G = AB, \ [A, B] = 1,$$

A is a Z-group, and B is either cyclic or a Z-group.

PROOF. Let $P_1, P_2, \cdots, P_s$ be a minimal basis of G. At most one P_i lies in $Z(G)$, since that is cyclic. Thus if P_1 is not in the centre of G, there is at least one P_j with which it does not commute. Renumber the basis elements if necessary so that P_1 and P_2 do not commute. Then since $[P_1,P_2] \neq 1$, and G' is cyclic of order p, $[P_1,P_2]$ must generate G'. Thus for $i > 2$, $[P_1,P_i] = [P_1,P_2]^{\alpha_i}$, $[P_2,P_i] = [P_1,P_2]^{\beta_i}$ for some $0 \leqq \alpha_i, \beta_i < p$. Let $P_i' = P_1^{-\beta_i}P_2^{-\alpha_i}P_i$. It is easy to see that $P_1, P_2, P_3', \cdots, P_s'$ is a minimal basis of G, and P_1, P_2 both commute with all of $P_3', \cdots, P_s'$. Let A, B be the subgroups of G generated by P_1, P_2 and $P_3', \cdots, P_s'$ respectively. It is immediate that $G = AB$, $[A,B] = 1$, $r(A) = 2$ and $r(B) = s - 2$.

COROLLARY. *A finite p-group G is a Z-group if and only if it contains normal subgroups* $A_1, \cdots, A_t$, *which each have rank at most* 2 *and all proper factors abelian, such that*

$$G = A_1A_2 \cdots A_t, [A_i, A_j] = 1 \text{ for } i \neq j, \text{ and } Z(A_1) \subset Z(A_2) \subset \cdots \subset Z(A_t).$$

The proof follows simply, by induction on the rank of G, from Theorems 2 and 3.

Thus to classify all Z-p-groups we need only classify those Z-p-groups of rank 2. To do this we distinguish between odd and even primes.

We first investigate Z-2-groups of rank 2. We denote by G^n the subset $\{x^n; x \in G\}$, and by $\langle A \rangle$ the group, or subgroup, generated by the set A.

THEOREM 5. *Let G be a Z-2-group of rank* 2. *Then there are three possibilities*:

(i) $G \cong D_4$, *the dihedral group of order* 8.

(ii) $G \cong Q$, *the quaternion group of order* 8.

(iii) *G has order* $2^{r+1} > 8$, *and can be generated by elements* a, b *satisfying the relations* $a^{2^r} = b^2 = 1$, $b^{-1}ab = a^{2^{r-1}+1}$.

PROOF. Since any group of order $\leqq 4$ is abelian, it is clear that G must have order $\geqq 8$. The only non-abelian groups of order 8 are D_4 and Q, and it is easy to see that both of these are Z-groups. It remains thus to consider the case where G has order $2^{r+1} > 8$.

Now $G/Z(G)$ cannot be cyclic, therefore it is of rank 2 (since G is of rank 2). Further, $x^2 \in Z(G)$ for all $x \in G$, so that $G/Z(G)$ is elementary abelian, and so has order 4. This means that $Z(G)$ has order 2^{r-1}, which is strictly greater than 2, the order of G'. Thus $G' \subset [Z(G)]^2$. Therefore $G' \subset G^2$, so that G/G^2 exists and is elementary abelian, and so $G^2 \supset Z(G)$. Let z be a generator of $Z(G)$, and $a \in G$ such that $a^2 = z$; a is of order 2^r.

Let now c be any element of G not in $\langle a \rangle$. Since G has order 2^{r+1}, we must have $\langle a, c\rangle = G$. $c^2 \in Z(G)$, so that $c^2 = z^u = a^{2u}$ for some integer u. We distinguish two cases.

i) u is even, so that $a^u \in Z(G)$. In this case put $b = a^{-u}c$. Then $b^2 = 1$, and $G = \langle a, b\rangle$.

ii) u is odd. In this case put $b = a^{-u-2^{r-2}}c$. Again, $b^2 = 1$ and $G = \langle a, b\rangle$. In either case, note that $a^{-1}b^{-1}ab = a^{2^{r-1}}$, so that $b^{-1}ab = a^{2^{r-1}+1}$, as stated.

Before investigating the Z-p-groups of rank 2 for odd p, we require a preparatory

LEMMA. *Let G be a finite p-group with all its proper factors abelian. If $p > 2$, we have*

$$(xy)^p = x^p y^p$$

for all $x, y \in G$.

PROOF. We have $G' \subset Z(G)$, so that G is nilpotent of class $\leqq 2 < p$, and is therefore regular [1, p. 73]. But G' is of order 1 or p, so that $c^2 = 1$ for all $c \in G'$. The result follows.

COROLLARY 1. $G^p = \{x^p; x \in G\}$ *is a subgroup of* G.

COROLLARY 2. $G^p \subset Z(G)$.

PROOF. $[x^p, y] = x^{-p}y^{-1}x^p y = x^{-p}(y^{-1}xy)^p = [x, y]^p = 1$, for all $x, y \in G$.

And now

THEOREM 6. *Let G be a Z-p-group of rank 2, with p an odd prime. Suppose $|G| = p^{r+1}$. Then*

EITHER

G has generators a, b satisfying $a^p = b^p = [a, b]^p = 1$ (and then G is of order p^3)

OR

G has generators a, b satisfying $a^{p^r} = b^p = 1$, $b^{-1}ab = a^{1+p^{r-1}}$.

PROOF. By the lemma, G^p is a normal subgroup of G, so that either $G^p = 1$ or G/G^p is abelian.

Let us take first the case $G^p = 1$. Since G is of rank 2, $G = \langle a, b\rangle$ for some a, b, and we must have $a^p = b^p = 1$. Since $G' \subset Z(G)$, every element of G can be expressed in the form $a^r b^s [a, b]^t$, so that G is of order at most p^3. Since G is not abelian, $[a, b]$ must be of order p, and G of p^3.

If, on the other hand, $G^p \neq 1$, then G/G^p is elementary abelian of rank 2, therefore of order p^2. But $G/Z(G)$ is of order at least p^2, since it cannot be cyclic, and $G^p \subset Z(G)$, so that we must have $G^p = Z(G)$. Let $Z(G) = \langle a^p \rangle$, and let c be any element of G not in $\langle a \rangle$. $c^p \in Z(G)$, so that $c^p = a^{pt}$ for some t. Thus if $b' = a^{-t}c$, we have $b'^p = 1$. Further, $[a, b'] \in \langle a^{p^{r-1}} \rangle$, so that $[a, b'] = a^{p^{r-1}s}$ for some $s \not\equiv 0$ (p). Now choose u so that $su \equiv 1$ (p), and put $b = b'^u$. Then $G = \langle a, b \rangle$, $a^{p^r} = b^p = 1$, and $[a, b] = a^{p^{r-1}}$.

REFERENCE

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